Rouen, France; KEITH EKBLAW, Walla Walla, WA, USA; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; MOUBINOOL OMARJEE, Paris, France; PEDRO HENRIQUE O. PANTOJA, student, UFRN, Brazil; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Bataille, Furdui, Janous, and Mortici showed that the result holds for all $p > 0$. Many solvers used the Stolz–Cesàro Theorem in their solutions; in particular, Janous gave the generalization that if $x_n \to \alpha$, $y_n \to \infty$, and the sequence $\{y_n\}$ is eventually monotonic, then $(x_1y_1 + x_2y_2 + \cdots + x_ny_n)/y_n \to \alpha$.

3470. [2009 : 396, 399] *Proposed by Mihaela Blanariu, Columbia College Chicago, Chicago, IL, USA.*

Let $p \geq 2$ be a real number. Find the limit

$$\lim_{n\to\infty} \frac{1 + \left(\sqrt{2!}\right)^p + \left(\sqrt[3]{3!}\right)^p + \cdots + \left(\sqrt[n]{n!}\right)^p}{n^{p+1}}.$$

Solution by Arkady Alt, San Jose, CA, USA.

Since

$$\left(\frac{n+1}{e}\right)^n \;<\; n! \;<\; (n+1)\left(\frac{n+1}{e}\right)^n,$$

then

$$\frac{n+2}{en} \;<\; \frac{\sqrt[n+1]{(n+1)!}}{n} \;<\; \sqrt[n+1]{n+2}\left(\frac{n+2}{en}\right).$$

Therefore, by the Squeeze Theorem for limits, we have that

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{n} \;=\; \frac{1}{e}.$$

Let $S_n = 1+(\sqrt{2!})^p+(\sqrt[3]{3!})^p+\cdots+(\sqrt[n]{n!})^p$ and $a_n = n^{p+1}$. According to the Stolz–Cesàro Theorem, if $\{a_n\}$ is an increasing sequence such that $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} \dfrac{S_{n+1} - S_n}{a_{n+1} - a_n}$ exists, then $\lim\limits_{n\to\infty} \dfrac{S_n}{a_n} = \lim\limits_{n\to\infty} \dfrac{S_{n+1} - S_n}{a_{n+1} - a_n}$. We have

$$\frac{S_{n+1} - S_n}{a_{n+1} - a_n} \;=\; \frac{\left(\sqrt[n+1]{(n+1)!}\right)^p}{(n+1)^{p+1} - n^{p+1}}$$

$$= \;\frac{\left(\sqrt[n+1]{(n+1)!}\right)^p}{n^p} \cdot \frac{1}{n\left(\left(1+\frac{1}{n}\right)^{p+1} - 1\right)}.$$

Since $\lim_{n\to\infty} n\left(\left(1+\frac{1}{n}\right)^{p+1}-1\right)=p+1$ and $\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n}=\frac{1}{e}$, then

$$\lim_{n\to\infty}\frac{S_n}{a_n} \;=\; \frac{1}{(p+1)e^p}$$

is the desired limit.

Also solved by GEORGE APOSTOLOPOULOS, Messolonghi, Greece; MICHEL BATAILLE, Rouen, France; OVIDIU FURDUI, Campia Turzii, Cluj, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; MOUBINOOL OMARJEE, Paris, France; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; JOEL SCHLOSBERG, Bayside, NY, USA; ALBERT STADLER, Herrliberg, Switzerland; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. One incorrect solution was submitted.

3471. [2009 : 396, 399] *Proposed by Cătălin Barbu, Bacău, Romania.*

Let ABC be an acute triangle and M, N, P be the midpoints of the minor arcs BC, CA, AB; respectively. If $[XYZ]$ denotes the area of triangle XYZ, prove that $[MBC]+[NCA]+[PAB]\geq s(3r-R)$, where s, r, and R are the semiperimeter, the inradius, and the circumradius of triangle ABC, respectively.

Solution by Edmund Swylan, Riga, Latvia, expanded by the editor.

The circumcentre O of ABC lies inside it, since the triangle is acute. Let a, b, c be the lengths of the sides opposite A, B, C, respectively, and let $[XYZW]$ denote the area of quadrilateral $XYZW$. Then

$$\begin{aligned}
&[ABC]+[MBC]+[NCA]+[PAB]\\
&\qquad = \;[MBOC]+[NAOC]+[PAOB]\\
&\qquad = \;\frac{1}{2}aR+\frac{1}{2}bR+\frac{1}{2}cR \;=\; sR\,.
\end{aligned}$$

From $[ABC]=rs$ we get $[MBC]+[NCA]+[PAB]=s(R-r)$, so the inequality to be proved is equivalent to $R\geq 2r$, which is just Euler's inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MARIAN DINCĂ, Bucharest, Romania; OLIVER GEUPEL, Brühl, NRW, Germany; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; THANOS MAGKOS, 3rd High School of Kozani, Kozani, Greece; CRISTINEL MORTICI, Valahia University of Târgovişte, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer. One incorrect solution was submitted.